Monday cup #4 – Solution

Posted on: April 15, 2019
Due on: April 21, 2019

Problem

Four distinct integers are to be chosen from the positive integers 1; 2; 3; 4; 5; 6 and 7. How many different selections are possible so the sum of the four positive integers is even?

Solution

We could look at every possible selection of four distinct numbers from the list, determine the sum of each selection and then count the number of selections for which the sum is even. There are 35 different selections to examine. A justification of this number is provided on the second page of this solution. This would not be an efficient approach!

We will make two simple observations. First, when even numbers are added together the sum is always even. And second, in order to produce an even sum using odd numbers, an even number of odd numbers is required in the sum. We will use these observations to break the problem into cases in which the sum is even. There are three cases to consider.

1. **No Odd Numbers are Selected**

   Since there are only three even numbers, namely 2, 4, and 6, it is not possible to select only even numbers. Therefore, there are no selections in which there are no odd numbers.

2. **Exactly Two Odd Numbers are Selected**

   There are four choices for the first odd number. For each of these four choices, there are three choices for the second number producing $4 \times 3 = 12$ choices for two odd numbers. However, each choice is counted twice. For example, 1 could be selected first and 3 could be chosen second or 3 could be selected first and 1 could be chosen second. Therefore, there are only $12 \div 2 = 6$ selections of two odd numbers. They are $\{1,3\}$, $\{1,5\}$, $\{1,7\}$, $\{3,5\}$, $\{3,7\}$, and $\{5,7\}$. For each of the 6 possible selections of two distinct odd numbers, we need to select two even numbers from the three even numbers in the list. We could use a similar argument to the selection of the two odd numbers or simply list the (three) possibilities: {2, 4}, {2, 6}, and {4, 6}. Therefore, there are $6 \times 3 = 18$ selections of four distinct numbers in which exactly two of the numbers are odd.

3. Exactly Four Odd Numbers are Selected

Since there are only four odd numbers in the list to choose from, there is only one way to select four distinct odd numbers from the list.

We have considered every possible case in which the selection produces an even sum. Therefore, there are $0 + 18 + 1 = 19$ selections of four distinct numbers from the list such that the sum is even.

There were correct solutions from Dato Tavdgiridze (Georgia, the country), Nika Darsalia (Georgia, the country) and Gela Tsetskhladze (Georgia, the country). The prize was split between Tavdgiridze and Tsetskhladze

Rules

1. Anyone is eligible to participate. Each solution is to be the work of one individual without any input from faculty or others. An answer must be accompanied by appropriate justifications to be considered correct.
2. The solution is to be submitted with the solver's name, email, year in school (if applicable), local phone number, and local address. If you are submitting this for possible credit in a class, include your class number and instructors name.
3. The solution is to be typed or legibly written. Solutions must be submitted to the by 2 p.m. on the due date.
4. Entries will be graded on clarity of exposition and elegance of solution. An award of GEL10 will be given for the best correct solution. In the case of a two-way tie, the award will be split. If there are more than two best solutions, a drawing will be held to determine two award winners.
5. Graduate students, faculty, and members of the general public are encouraged to submit solutions, but they will not be considered.

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Solution for this problem can be submitted proveweek@gmail.com