Monday cup #22- Solution

Posted on: August, 26,2019
Dueon: September,*01*,201 9

Problem

Question: **A "Kaxaberi" store clerk regards as a lucky number any** positive integer that cannot be **obtained as a sum of sevens and/or elevens. For instance, 13** and 20 are lucky numbers, while **14 = 7 + 7, 25 = 7 + 7 + 11, 40 = 7 + 11 + 11 + 11, and** 44 = 11 + 11 + 11 + 11 are not. **Find (with proof) the largest lucky number.**

Solution to Problem:

A "Kaxaberi" store clerk regards as a lucky number any positive integer that cannot be obtained as a sum of sevens and/or elevens. For instance, 13 and 20 are lucky numbers, while 14 = 7 + 7, 25 = 7 + 7 + 11,
40 = 7 + 11 + 11 + 11, and 44 = 11 + 11 + 11 + 11 are not.

Solution. We claim that the largest lucky number is 59.

If N is a lucky integer, the equation

(1) $$7x + 11y = N$$

has no solution where x, y are non-negative integers.

However, we claim that this equation has solutions if we allow x, y to be arbitrary integers, positive or negative and we find all such solutions. Notice first that $7(3) + 11(2) = 1$, so $x' = -3N$, $y' = 2N$ is a solution of the above equation, i.e. $7x' + 11y' = N$. Subtracting, we see that $7(x - x') + 11(y - y') = 0$. Since 7 and 11 are prime to each other, this implies that there is an integer t such that $x - x' = 11t$ and $y - y' = -7t$.

Therefore the general solution in integers of the equation $7x + 11y = N$ is given by

(2) $$x = -3N + 11t, \quad y = 2N - 7t$$

where t is an arbitrary integer.

Further, there is a *unique* solution (x_0, y_0) of (1) with $0 \leq y_0 \leq 6$. Indeed, we can keep increasing t by one unit till we have $0 \leq y_0 \leq 6$, and at this point we cannot increase t any further.

For this special solution (x_0, y_0) with $0 \leq y_0 \leq 6$, we must have $x_0 < 0$, since otherwise N would be unlucky as $N = 7\mathrm{x}x_0 + 11\mathrm{x}y_0$ with $-x_0 \leq 0$, $-y_0 \leq 0$. Therefore the largest lucky integer corresponds to $x_0 = 1$ and $y_0 = 6$
i.e the largest lucky integer is

$$7(-1) + 11(6) = 59.$$

There was no correct solution to problem 22

Rules

1. Anyone is eligible to participate. Each solution is to be the work of one individual without any input from faculty or others. An answer must be accompanied by appropriate justifications to be considered correct.
2. The solution is to be submitted with the solver’s name, email, year in school (if applicable), local phone number, and local address. If you are submitting this for possible credit in a class, include your class number and instructors name.
3. The solution is to be typed or legibly written. Solutions must be submitted to the by 2 p.m. on the due date.
4. Entries will be graded on clarity of exposition and elegance of solution. An award of GEL10 will be given for the best correct solution. In
the case of a two-way tie, the award will be split. If there are more than two best solutions, a drawing will be held to determine two award winners.
5. Graduate students, faculty, and members of the general public are encouraged to submit solutions, but they will not be considered.

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Solution for this problem can be submitted proveweek@gmail.com